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Dèbes - Emsalem's Theorem

Theorem 1.

Let $K < F$ be a finite Galois extension, X a projective algebraic variety defined over F, and $K = M_{F/K}(X)$. If the birational automorphisms group of X defined over F, $Aut(X)$, is a finite group and $P : X \to X/Aut(X)$ the Galois covering with cover group $Aut(X)$. Then:

(1) *Exists a projective algebraic variety $B \simeq X/Aut(X)$, defined over K (called canonic model of $X/Aut(X)$), and an isomorphism $R : X/Aut(X) \to B$ defined over F, such that, if $Q = R \circ P$, then $M_{F/K}(Q : X \to B) = K$.*

(2) *Moreover, if we can find a point in $B - B_Q$, where B_Q is de set of all critical values of Q, which are K-rational, then K is a definition field of X.*

Proof. First of all, note that:

(1)
$$\begin{aligned} \mathrm{Aut}(X^\sigma) &= \{h : X^\sigma \to X^\sigma \ / \ h \in \mathrm{Aut}(X^\sigma)\} \\ &= \{h^\sigma : X \to X \ / \ h^\sigma \in (\mathrm{Aut}(X))^\sigma\} \\ &= (\mathrm{Aut}(X))^\sigma \end{aligned}$$

(2) Let $p := [x_0 : x_1 : ... : x_n]$ be a point in X, then $\sigma(p) := [\sigma(x_0) : \sigma(x_1) : ... : \sigma(x_n)] \in X^\sigma$. Observe the following commutative diagram

$$\begin{array}{ccc} X^\sigma & \xrightarrow{\varphi} & (X/\mathrm{Aut}(X))^\sigma \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \psi} & \\ X^\sigma/\mathrm{Aut}(X^\sigma) & & \end{array}$$

where $\pi(\sigma(p)) = [\sigma(p)]_{\mathrm{Aut}(X^\sigma)}$, $\varphi(\sigma(p)) = \sigma([p]_{\mathrm{Aut}(X)})$ and $\psi(\sigma[p]_{\mathrm{Aut}(X)}) = [\sigma(p)]_{\mathrm{Aut}(X^\sigma)}$. This implies that $X^\sigma/\mathrm{Aut}(X^\sigma)$ and $(X/\mathrm{Aut}(X))^\sigma$ are canonically isomorphic.

Since $K = M_{F/K}(X)$, by definition, for each $\sigma \in \mathrm{Aut}(F/K)$, exists a birational isomorphism

$$f_\sigma : X \to X^\sigma$$

defined over F. By other hand, necessarily exists a birational isomorphism (only determined by σ)

$$g_\sigma : X/\mathrm{Aut}(X) \to (X/\mathrm{Aut}(X))^\sigma$$

such that $g_\sigma \circ P = P^\sigma \circ f_\sigma$, as shown in the diagram below:

$$\begin{array}{ccc} X & \xrightarrow{f_\sigma} & X^\sigma \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P^\sigma} \\ X/\mathrm{Aut}(X) & \xrightarrow{g_\sigma} & (X/\mathrm{Aut}(X))^\sigma \end{array}$$

Note that, for every $\sigma, \tau \in \mathrm{Aut}(F/K)$, holds $f_\tau^\sigma \circ f_\sigma \circ f_{\sigma\tau}^{-1} \in \mathrm{Aut}(X^{\tau\sigma})$. Then,

$$\begin{array}{rrcl} & P^{\sigma\tau} & = & P^{\sigma\tau} \circ f_\tau^\sigma \circ f_\sigma \circ f_{\sigma\tau}^{-1} \\ \Leftrightarrow & P^{\sigma\tau} \circ f_{\sigma\tau} & = & P^{\sigma\tau} \circ f_\tau^\sigma \circ f_\sigma \circ (f_{\sigma\tau}^{-1} \circ f_{\sigma\tau}) \\ \Leftrightarrow & g_{\sigma\tau} \circ P & = & P^{\sigma\tau} \circ f_\tau^\sigma \circ f_\sigma \\ & & = & (P^\tau \circ f_\tau)^\sigma \circ f_\sigma \\ & & = & (g_\tau \circ P)^\sigma \circ f_\sigma \\ & & = & g_\tau^\sigma \circ P^\sigma \circ f_\sigma \\ & & = & g_\tau^\sigma \circ g_\sigma \circ P \end{array}$$

This means, $g_{\sigma\tau} \circ P = g_\tau^\sigma \circ g_\sigma \circ P$, which is equivalent to

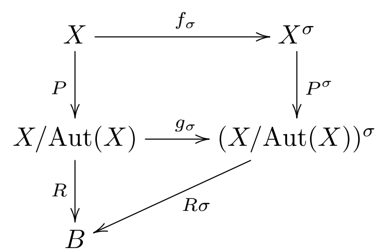
$$g_{\sigma\tau} = g\tau^\sigma \circ g_\sigma$$

Then, from Weil's theorem, we have that exists a variety B defined over K and a birational isomorphism

$$R : X/\mathrm{Aut}(X) \to B$$

defined over F, such that $R = R^\sigma \circ f_\sigma$, for every $\sigma \in \mathrm{Aut}(F/K)$.

The following diagram shows what is happening



Let $Q = R \circ P : X \to B$. For each $\sigma \in \mathrm{Aut}(F/K)$, we have $Q = Q^\sigma \circ f_\sigma$. Indeed:

$$\begin{array}{rcl} Q^\sigma \circ f_\sigma & = & R^\sigma \circ P^\sigma \circ f_\sigma \\ & = & R^\sigma \circ g_\sigma \circ P \\ & = & R \circ P \\ & = & Q \end{array}$$

Now, by definition, $M_{F/K)}(X) < M_{F/K}(Q : X \to B)$, but $Q = Q^\sigma \circ f_\sigma$ implies that for every $\sigma \in \mathrm{Aut}(F/K) < F_K(Q : X \to B)$ as a subgroup, taking the fixed field of these groups we have $M_{F/K}(Q : X \to B) < M_{F/K}(X)$ as a subfield, thereby $M(Q : X \to B) = M_{F/K}(X) = F$.

To finish the proof, suppose that exists a K- rational point $r \in B - B_Q$, and let $p \in X$ such that, $Q(p) = r$.
If $\sigma \in \mathrm{Aut}(X)$, then $\sigma(P) \in X^\sigma$ and $Q^\sigma(\sigma(p)) = \sigma(Q(p)) = \sigma(r) = r$. Thus, exists $h_\sigma \in \mathrm{Aut}(X)$, such that $(f_\sigma \circ h_\sigma)(p) = \sigma(p)$.
Let $t_\sigma = f_\sigma \circ h_\sigma$, then t_σ is an isomorphism which is only determined, because if there was another isomorphism, say $t : X \to X^\sigma$, such that $t(p) = \sigma(p)$, then $h = t^{-1} \circ t_\sigma \in \mathrm{Aut}(X)$, where $h(p) = p$, with $h = t^{-1} \circ t_\sigma$. Since, $r \in B - B_Q$, r is not a critic value of Q, so $h = I$ and then $t = t_\sigma$.

By the uniqueness of the choice of t_σ, ensures that the family $\{t_\sigma : \sigma \in \mathrm{Aut}(F/K)\}$ satisfies the conditions of Weil's theorem.

□

Remark 2. In the proof of the above theorem, we have an isomorphism $R : X/\mathrm{Aut}(X) \to B$ satisfying the property that for each $\sigma \in \mathrm{Aut}(F/K)$, holds $R = R^\sigma \circ g_\sigma$.
Searching a K-rational point $r \in B - B_Q$ is equivalent to search a point $s \in X/\mathrm{Aut}(X) - B_P$, such that $\sigma(R(s)) = R(s)$, for every $\sigma \in \mathrm{Aut}(F/K)$. Since $\sigma(R(s)) = R^\sigma(\sigma(s))$, this is equivalent to search a point $s \in X/\mathrm{Aut}(X) - B_p$ such that $g_\sigma(s) = \sigma(s)$, for each $\sigma \in \mathrm{Aut}(F/K)$. That means that for searching of K-rational points in B is not necessary to know B.