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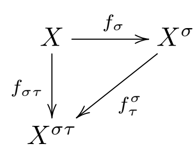
Weil's Theorem

Theorem 1.

Let $K < F$ be a finite Galois extension, $X \subset \mathbb{P}^n_F$ a projective algebraic variety defined over F and $K = M_{F/K}(X)$. Suppose that for each $\sigma \in F_K(X) = Gal(F/K)$ exists a birational application $f_\sigma : X \to X^\sigma$ defined over F such that, for each pair $\sigma, \tau \in Gal(F/K)$, $f_{\sigma\tau} = f_\tau^\sigma \circ f_\sigma$, that is, the following diagram commutes



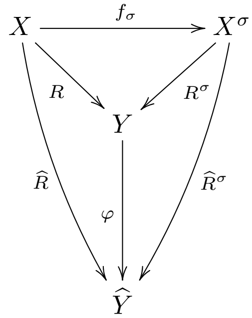
Then:

(1) *There exists a projective algebraic variety Y, defined over K, and there exists a birational application $R : X \to Y$ defined over F such that, for every $\sigma \in Gal(F/K)$ it holds the equality $R^\sigma \circ f_\sigma = R$. Moreover, if every f_σ is a biregular, then we can assume R to be a biregular isomorphism.*

(2) *If the pair $(\widehat{R}, \widehat{Y})$ is another solution for the above problem, then exists a birational isomorphism $\varphi : Y \to \widehat{Y}$ defined over F such that, $\widehat{R} = \varphi \circ R$.*

Proof. **(Uniqueness)** Suppose that $R : X \to Y \simeq \widehat{R} : X \to \widehat{Y}$ are defined over F such that, for every $\sigma \in \mathrm{Gal}(F/K)$ holds $R^\sigma \circ f_\sigma = R$ and $\widehat{R}^\sigma \circ f_\sigma = \widehat{R}$.
Let $\varphi : Y \to \widehat{Y}$ such that, $\varphi = \widehat{R} \circ R^{-1}$.

The following diagram shows how the arrows are



It is easy to see that φ is already defined over F. Moreover, φ is defined over K. Indeed:

$$\begin{array}{rcl}\varphi^{\sigma}=(\widehat{R}\circ R^{-1})^{\sigma} & = & \widehat{R}^{\sigma}\circ (R^{\sigma})^{-1}\\ & = & \widehat{R}^{\sigma}\circ (f_{\sigma}\circ f_{\sigma}^{-1})\circ (R^{\sigma})^{-1}\\ & = & \widehat{R}^{\sigma}\circ f_{\sigma}\circ (R^{\sigma}\circ f_{\sigma})^{-1}\\ & = & \widehat{R}\circ R^{-1}\\ & = & \varphi\end{array}$$

(Existence for the birational case)

Let $F(X)$ be the rational function field of X defined over F, which is finitely generated over F.
If $\sigma \in \mathrm{Aut}(F/K)$ and $\phi \in F(X)$, then

$$\phi^{\sigma}=\sigma\circ\phi\circ\sigma^{-1}\in F(X^{\sigma})$$

Since, $f_{\sigma}: X \to X^{\sigma}$ is birational, we can consider $\phi^{\sigma}\circ f_{\sigma}\in F(X)$.

It defines

$$\begin{array}{rcccl}\sigma^{*} & : & F(X) & \to & F(X)\\ & & \phi & \mapsto & \sigma^{*}(\phi)=\phi^{\sigma}\circ f_{\sigma}\end{array}$$

Note that, $\sigma^{*}\in \mathrm{Aut}(F(X)/K)$, and $f_e = I_X$ implies $e^{*} = I_{\mathrm{Gal}(F(X)/K)}$.

We consider the following lemma.

LEMMA 2. *The function*

$$\begin{array}{rcccl}\Phi & : & Gal(F/K) & \to & Gal(F(X)/K)\\ & & \sigma & \mapsto & \sigma^{*}\end{array}$$

is a injective group homomorphism.

PROOF.

(1) Φ is a group homomorphism.

$$\begin{array}{rcl}(\Phi(\sigma)\circ\Phi(\tau))(\phi) & = & \Phi(\sigma)(\Phi(\tau)(\phi))\\ & = & \Phi(\sigma)(\tau^{*}(\phi))\\ & = & \Phi(\sigma)(\phi^{\tau}\circ f_{\tau})\\ & = & \sigma^{*}(\phi^{\tau}\circ f_{\tau})\\ & = & (\phi^{\tau}\circ f_{\tau})^{\sigma}\circ f_{\sigma}\\ & = & (\phi^{\tau})^{\sigma}\circ f_{\tau}^{\sigma}\circ f_{\sigma}\\ & = & \phi^{\sigma\tau}\circ f_{\sigma\tau}\\ & = & \Phi(\sigma\tau)(\phi)\end{array}$$

(2) Φ is injective.

$$\begin{array}{rcl}\mathrm{Ker}(\Phi) & = & \{\sigma\in \mathrm{Gal}(F/K) : \Phi(\sigma)=I_{\mathrm{Gal}(F(X)/K)}\}\\ & = & \{\sigma\in \mathrm{Gal}(F/K) : \phi^{\sigma}\circ f_{\sigma}=\phi,\ \forall\phi\in F(X)\}\\ & = & \{\sigma\in \mathrm{Gal}(F/K) : \sigma\circ\phi\circ\sigma^{-1}\circ f_{\sigma}=\phi,\ \forall\phi\in F(X)\}\end{array}$$

By taking $\phi = a \in F$ constant, we obtain that if $\sigma \in \mathrm{Ker}(\Phi)$, then $\sigma(a) = a$. It follows that $\{I_{\mathrm{Gal}(F/K)}\}$.
□

From the above lemma, $\Gamma^{*} = \Phi(\mathrm{Gal}(F/K))$ is a finite group isomorphic to $\Gamma = \mathrm{Gal}(F/K)$.
Since Γ^{*} is a subgroup of $\mathrm{Gal}(F(X)/K)$, we can consider its fixed field, say $\mathbb{F}$, which satisfies the following properties:

i) $K < \mathbb{F} < F(X)$
ii) $\mathbb{F} < F(X)$ is a finite Galois extension.
iii) $\mathbb{F}$ is finitely generated over K.

Suppose that $\mathbb{F} =< \psi_1, ..., \psi_s >$, with $\psi_j \in F(X)$, and take the homomorphism

$$\begin{array}{rccc} \Theta & : & K[x_1,...,x_s] & \rightarrow & \mathbb{F} \\ & & x_j & \mapsto & \psi_j \end{array}$$

It is a fact that:

(1) $\mathrm{Ker}(\Theta)$ defines an (affine) algebraic variety X_0 defined over K and $K[X_0] = \frac{K[x_1,...,x_s]}{\mathrm{Ker}(\Theta)}$, and Θ induces a field isomorphism between $K(X_0)$ and $\mathbb{F}$.

(2) $F(X_0) = F(X)$. In fact, since $\mathbb{F} < F(X)$ and $\mathbb{F} \cong K(X_0)$, then $F(X_0) < F(X)$. On the other hand, if $H^* = \{\sigma^* \in \mathrm{Gal}(F(X)/K) : \sigma \in \Gamma, \sigma(\alpha) = \alpha, \ \forall \alpha \in F, \sigma^*(\psi_j) = \psi_j\}$, then $\mathrm{Fix}(H^*) = F(X_0)$. Next, we note that, if $\sigma^* \in H^*$,then $\sigma = e$, and $\sigma^* = I$. In particular, $\mathrm{Fix}(H^*) = F(X)$. Therefore, $F(X) = F(X_0)$.

As a consequence, $X \simeq X_0$ over F and there is an isomorphisms $R : X \rightarrow X_0$, defined over F, so that, for every $\phi \in L(X)$, it holds that

$$(*) \quad \phi = Q(\psi_1, ..., \psi_s) \circ R$$

where Q is rational in s variables and defined over L. Moreover, $\phi \in \mathbb{F}$ if and only if Q has its coefficients in K.

In this way, if $\phi \in \mathbb{F}$ and $\sigma \in \Gamma$, then we may write $\phi = \psi \circ R$, where $\psi^\sigma = \psi$. So, we have

$$\psi \circ R = \phi = \sigma^*(\phi) = \sigma^*(\psi \circ R) = (\psi \circ R)^\sigma \circ f_\sigma = \psi^\sigma \circ R^\sigma \circ f_\sigma = \psi \circ R^\sigma \circ f_\sigma.$$

From the above, we may deduce that $R = R^\sigma \circ f_\sigma$.

$\square$